Noisy Maps near Crises *

Peter Reimann
L.U.C., 3590 Diepenbeek, Belgium

Z. Naturforsch. **49a**, 1248–1250 (1994); received October 12, 1994

We address the escape rate for single-humped maps near a boundary crisis multiplicatively coupled to weak uncorrelated noise. A scaling law for the rate is derived predicting a stabilization of deterministic transient chaos by noise. Generalizations to maps near interior crises and band merging points are given.

A large variety of problems arising in very different scientific areas can be approximately described by low dimensional nonlinear dynamics [1]. Of particular interest are sudden qualitative changes that can occur in such systems, for instance bifurcations or crises [2]. Typically, in such a low dimensional description small perturbations arising from a large number of fast variables are neglected. However, the influence of this "environment" should be included in a more realistic model in the form of weak noise, in particular close to the above-mentioned sudden qualitative changes of the deterministic dynamics.

We consider a one-dimensional dynamics in discrete time n in the presence of weak multiplicative noise:

$$x_{n+1} = f(x_n) + \sigma g(x_n)\,\xi_n . \qquad (1)$$

Here, $f(x)$ is a single-humped map of the real axis with a maximum of order $z > 0$ at $x = x^*$,

$$f(x) = 1 + \Delta - b\,|x - x^*|^z + o(|x - x^*|^z), \qquad (2)$$

where Δ is a small parameter, $-1 \ll \Delta \ll 1$, and $b > 0$. Further, $f(x)$ is strictly monotonical on both sides of the maximum x^* and continuously differentiable everywhere with the exception of $x = x^*$ for $z \le 1$. The x-scale is chosen such that $f(x)$ has an unstable fixed point at $x = 0$, $f(0) = 0$, and a second zero at $x = 1$, $f(1) = 0$, implying $f'(0) > 1$, $f'(1) < 0$, and $0 < x^* < 1$. Thus, at $\Delta = 0$ the map undergoes a boundary crisis [2] and shows fully developed chaos [3]. A well known example with $z = 2$ is the logistic map

$$f(x) = 4(1 + \Delta)\,x\,(1 - x). \qquad (3)$$

* Presented at the Workshop on Thermodynamic Formalism Lalvin/Engadin, Switzerland 23–26 April 1994.

Reprint requests to P. Reimann.

The noise-strength σ in (1) is required to be small, $0 \le \sigma \ll 1$, and the noise-coupling function $g(x)$ to be bounded on $\mathbb{R}$ and continuous at 0, x^*, and 1. The noise ξ_n is given by independent, identically distributed random numbers with a probability distribution $P(\xi)$ that decreases faster than $1/|\xi|^{1+1/z}$ for large $|\xi|$.

In order to determine the probability density of the stochastic process (1) in the quasi-stationary state we developed a novel perturbation method [4] in the two small parameters Δ and σ about the stationary probability density $\varrho(x)$ at fully developed chaos $\Delta = 0$ [3] in the absence of noise $\sigma = 0$. This method has certain similarities to singular perturbation theory for differential equations. For $z > 1$ and $\Delta < 0$ the deterministic map $f(x)$ shows periodic windows [1] giving rise to considerable difficulties for any perturbation theory about $\Delta = \sigma = 0$ [5]. We succeeded to derive the necessary and sufficient condition

$$\sigma \gg |\Delta|^{\frac{z}{z-1}} \quad (z > 1,\ \Delta < 0) \qquad (4)$$

for the validity of our method [4]. From the quasi-stationary probability density one finds the escape rate k out of the unit interval $[0, 1]$

$$k = \varrho(x^*)\left(\frac{\sigma}{b}\right)^{\frac{1}{z}} F\left(\frac{\Delta}{\sigma}\right), \quad F(x) = 2\int_0^\infty \mathrm{d}y\, y^{\frac{1}{z}}\, h_\infty(y - x). \qquad (5)$$

The function $h_\infty(x)$ is given as the $l \to \infty$ limit of

$$h_l(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{ikx} \prod_{m=0}^{l} \tilde{P}(T_m k), \quad T_m = \frac{g(f^m_{\Delta=0}(x^*))}{\frac{\mathrm{d}}{\mathrm{d}x} f^m_{\Delta=0}(x^*)}, \qquad (6)$$

where $\tilde{P}(k) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, e^{-ik\xi} P(\xi)$ is the Fourier transform of the noise distribution. For the single-humped maps at the boundary crisis $f_{\Delta=0}(x)$ considered here one has $T_0 = g(x^*)$, $T_1 = g(1)/f'(1)$, $T_m = g(0)/[f'(1)\,f'(0)^{m-1}]$, $m \ge 2$.

0932-0784 / 94 / 1200-1248 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72027 Tübingen

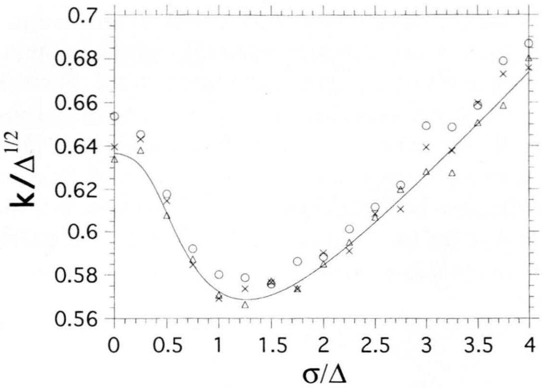

Fig. 1. Escape rate k versus noise-strength σ for the logistic map (3) with additive Gaussian noise $g(x) \equiv 1$, $P(\xi) = (2\pi)^{-1/2} e^{-\xi^2/2}$. The solid line is the theoretical prediction (5), (6) with $\varrho(x^*) = 2/\pi$ [1]. The symbols are results from numerical simulations of (1) for $\Delta = 10^{-5}$ (circles), $\Delta = 10^6$ (crosses), and $\Delta = 10^{-7}$ (triangles). The statistical uncertainty is about 1%. For $\Delta = 10^{-5}$ the numerical results are systematically slightly above the theoretical line which clearly is a finite Δ and σ effect.

Our central rate formula (5) has the form of a scaling law [6, 7]. It becomes asymptotically exact for small Δ and σ respecting the extra condition (4) and indeed compares very well with the numerical simulations shown in Figure 1. Apart from $\varrho(x^*)$, which is globally dependent on $f(x)$, only local properties of the map $f(x)$ and the noise-coupling function $g(x)$ at 0, x^*, and 1 enter the rate (5). For $\Delta \gg \sigma$ the correct deterministic limit $k = 2\varrho(x^*)(\Delta/b)^{1/z}$ is recovered. In previous investigations, rate formulas similar to (5) have been derived, but for less general maps $f(x)$, noise coupling functions $g(x)$, and noises ξ_n in (1), and with the approximation $h_\infty(x) = h_0(x)$ or $h_\infty(x) = h_1(x)$ in (5), (6), see for example [6] or [7], respectively. One can easily find examples (1) for which these approximations agree very well or very badly with the asymptotically exact result (5), (6). The scaling law (5) can also be written under the form $k = 2\varrho(x^*)(|\Delta|/b)^{1/z}\, G(\sigma/\Delta)$, where the new scaling function $G(x)$ is related in an obvious way to $F(x)$. Closer inspection shows that on $\mathbb{R}_+$ the function $G(x)$ exhibits a global minimum at $x_{\min} > 0$ whenever the noise distribution is symmetric, $P(-\xi) = P(\xi)$, and $z > 1$. As a function of z, this minimum $x_{\min}$ is monotonically increasing, becoming 0 for $z \to 1$ and proportional to z for large z. The minimum value $G(x_{\min})$ monotonically decreases from 1 for $z = 1$ to 1/2 for $z \to \infty$. Thus, for any $\Delta > 0$ sufficiently small noise-strengths σ will lead to smaller rates k than in the absence of noise $\sigma = 0$, see Figure 1. In other words, the noise induces a stabilization of deterministic transient chaos [8]. The same effect occurs for sufficiently large z-values even if the noise distribution is no longer symmetric. A simple intuitive explanation is possible only in the particular case that all the T_m in (6) vanish for $m \geq 1$ [4].

As a particular example we first consider the symmetric Lévy distributions given by $\tilde{P}(k) = \exp\{-|k|^\mu\}$, $0 < \mu \leq 2$. For $\mu = 2$ and $\mu = 1$ one recovers Gaussian and Lorentz distributions $(P(\xi) = [\pi(1+\xi^2)]^{-1})$, respectively. From (6) one readily finds that $h_\infty(x) = P(x/U_\mu)/U_\mu$, where $U_\mu = \left(\sum_{m=0}^{\infty} |T_m|^\mu\right)^{1/\mu}$. Consequently, the scaling law (5) is equivalent to $k = \varrho(x^*)(U_\mu \sigma/b)^{1/z}\, \tilde{F}(\Delta/[U_\mu \sigma])$, where the new scaling function $\tilde{F}(x) = 2\int_0^\infty dy\, y^{1/z} P(y-x)$ is universal for any fixed z and μ.

Next we address exponentially distributed noise $P(\xi) = N\exp\{-|\xi|^\alpha\}$, $\alpha > 0$, where N is a normalization constant. For ordinary exponential ($\alpha = 1$), Gaussian ($\alpha = 2$), and confined ($\alpha = \infty$) noise a simple closed expression for $h_\infty(x)$ in (6) can be found [4]. Further, for asymptotically large x one can show that

$$h_\infty(x) = e^{-|x/A_\infty|^\alpha} \frac{\alpha}{2\Gamma(1/\alpha)A_\infty} \quad \text{for } 0<\alpha<1, \tag{7}$$

$$h_\infty(x) = e^{-|x/A_\infty|} \frac{1}{2A_\infty} \prod_{T_m^2 \neq A_\infty^2} \frac{A_\infty^2}{A_\infty^2 - T_m^2} \quad \text{for } \alpha=1, \tag{8}$$

$$h_\infty(x) = e^{-|x/A_\infty|^\alpha} |x|^{\beta(x)\ln|x|} \quad \text{for } \alpha>1, \tag{9}$$

where

$$A_\infty = \max_m |T_m| \quad \text{for } 0 < \alpha \leq 1,$$

$$A_\infty = \left(\sum_{m=0}^{\infty} |T_m|^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \quad \text{for } \alpha > 1,$$

and $\beta(x) \to (\alpha-1)(2-\alpha)/[4\ln f'(0)]$ for $x \to \infty$. From (7)–(9) one readily finds that the rate (5) is dominated by an Arrhenius-like factor $\exp\{-|\Delta/\sigma A_\infty|^\alpha\}$ in the deep precritical regime $\Delta \ll -\sigma$ (but still respecting (4)), in agreement with [9, 10]. The pre-exponential factor of the rate is algebraic in $|\Delta|$ and σ for $\alpha \leq 1$ and $\alpha = 2$, whereas for the remaining α-values the $|\Delta|$ and σ dependence is stronger than any power law, similarly as in (9).

Our method to determine the quasi-stationary probability density and the rate can be generalized to single-humped maps near an interior crisis [2] or a band merging point [1]. In this case, a suitable iterate of the map, say $f^p(x)$, exhibits a boundary crisis when restricted to any of p properly chosen disjoint subintervals I_i, $i = 1, \ldots, p$, of [0, 1]. Equation (2) generalizes to $f(x) = x_1 + \Delta - b|x - x^*|^z + \ldots$, where x_1 is mapped under $f^p(x)$ on the unstable periodic orbit which collides with the strange attractor at the crisis or band merging point. At $\Delta = 0$, the intervals I_i are mapped onto each other by $f(x)$ and onto themselves by $f^p(x)$. If one denotes by $p \cdot k$ the escape rate in the quasi-stationary state from I_i after p time steps, then exactly the same result (5), (6) for k is found, independent of the particular interval I_i [4]. Further generalizations to multiple-humped maps and exponentially correlated Gaussian noise are possible [4].

Financial support by the Swiss National Science Foundation, the Freiwillige Akademische Gesellschaft, Basel, and the Program on Inter-University Attraction Poles of the Belgian Government is gratefully acknowledged.

[1] H. G. Schuster, "Deterministic Chaos", VCH Verlag, Weinheim 1988.
[2] C. Grebogi, E. Ott, and Y. A. Yorke, Physica **7D,** 181 (1983).
[3] G. Györgyi and P. Szépfalusy, J. Stat. Phys. **34,** 451 (1984).
[4] P. Reimann, submitted to J. Stat. Phys.
[5] S. Grossmann, Z. Phys. **B57,** 77 (1984).
[6] F. T. Arecchi, R. Badii, and A. Politi, Phys. Lett. **103 A,** 3 (1984).
[7] J. C. Sommerer, E. Ott, and C. Grebogi, Phys. Rev. A **43,** 1754 (1991).
[8] M. Franaszek, Phys. Rev. A **44,** 4065 (1991).
[9] P. D. Beale, Phys. Rev. A **40,** 3998 (1989).
[10] A. Hamm, T. Tél, and R. Graham, Phys. Lett. **185 A,** 313 (1994).